

Tail asymptotics for diffusion processes, with applications to local volatility and CEV-Heston models

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Abstract

We characterize the tail asymptotics for a pure diffusion process (Theorem 1.2), and a time-dependent local volatility model (Theorem 1.3), using results by Doss [D77] and Norris & Stroock [NS91] respectively. The asymptotics obtained are expressed in terms of a certain geodesic distance, and in the general case in terms of an Energy functional. We compare these two quantities qualitatively and numerically, by solving the Euler-Lagrange equation. We also describe the large-strike implied volatility smile asymptotics, using the right-tail-wing formula of Benaim & Friz [BF06 I]. We derive a similar result for the CEV process (Theorem 1.6), and a CEV process evaluated at independent stochastic time $\tau(\omega, t)$ (Theorem 2.1). The latter is applicable to the CEV-Heston model introduced by Atlan & Leblanc [AL05]. Finally, we show that if we wish to use an extended version of the Carr-Lee [CL04] methodology to infer the characteristic function of $\tau(t)$ from an observed single-maturity smile under the time-changed CEV model, then the tails of the distribution function of $\tau(t)$ must have sub-exponential behaviour.

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Introduction and literature review

Davies[Dav87] introduced a new technique, using a log Sobolev inequality, for obtaining sharp bounds for the heat kernel, in terms of the *geodesic distance* under the Riemmanian metric induced by the inverse of the second order coefficient of the diffusion (see Appendix). However, the operator governing the diffusion process had to be in the so-called *divergence* form

$$L = \nabla \cdot [a(x) \nabla] \quad (0.1)$$

and the coefficients had to be time-independent. Norris&Stroock[NS91] succeeded in removing these restrictions, but the bounds thus obtained were now expressed in terms of a more general *Energy functional* $E(t, x; u, y)$, which is the solution to a variational problem which is more difficult to compute than the Riemmanian distance function. We apply their result to a time-inhomogeneous local volatility model in Theorem1.3.

In the time-independent case, our approach (in one dimension) is simpler, and merely consists of bounding (or more precisely, “sandwiching”) the Itô functional $I : C[0, T] \rightarrow C[0, T]$, which maps the Brownian control process to the response i.e. the unique strong solution of the SDE, based on the procedure outlined in Doss[D77] (see section 1). We use this to characterize the tail behaviour of the distribution function of the marginal distributions for a general one-dimensional SDE, subject to certain regularity conditions, in terms of the geodesic distance (Theorem1.2). In general, the Itô map is only continuous (in the topology of uniform convergence) in the one-dimensional case (see e.g. chapter 2 of Lyons&Qian[LQ02]), Sussman[Suss78], Lamperti[Lam64] and page 188 in Dembo&Zeitouni[DemZei93]).

Benaim&Friz[BF06 I] used *regular variation* theory to show how the tail asymptotics of the log Stock price translate *directly* to the large-strike behaviour of the implied volatility smile. This tells us when the \limsup in Lee’s moment formula[Lee04] becomes a genuine limit. They also describe the smile asymptotics when all moments of the terminal Stock price exist, where the implied variance exhibits *sub-linear* behaviour in the wings. In their sequel paper, Benaim&Friz[BF06 II] develop criteria for establishing when the aforementioned \limsup is a limit, in the case when the moment generating function of the log Stock price is known. This is accomplished using Tauberian theorems, which look closer at the limiting behaviour of the log of the Stock price mgf around the critical value where the moment explosion occurs. In particular, they are able to characterize the behaviour of the transition densities (on logarithmic scale) of the Stock price and the integrated variance under the well known Heston model.

We apply the right-tail-wing formula to characterize the large-strike implied volatility smile asymptotics for a general time-inhomogenous local volatility model (Theorem 1.5), and for a CEV process 1.7. For a CEV process subordinated to an independent stochastic clock, we establish a one-to-one correspondence between certain exponential moment explosions for the law of the time-change, and the law of the terminal Stock price. This result can be specialized to

the CEV-Heston stochastic volatility model discussed in Atlan&Leblanc[AL05], and for this reason we advocate using this model over the popular SABR model introduced by Hagan et al.[HKLW02], because less is known about the tail behaviour of the transition pdf for this parametrization (see Remark 2.4)

In section 3, we discuss calibration issues for a time-changed diffusion model. We use Eq 2.6 to extend the Carr-Lee[CL04] methodology for reverse engineering the law of the time-change from the law of the composite Stock price process at a single time. We find that the distribution function of the time-change has to have sub-exponential tail behaviour or else we have to resort to analytic continuation.

1 Tail asymptotics for the transition densities of one-dimensional SDEs

1.1 Doss's theorem

We recall the following result from Doss[D77], which is given as Proposition 2.21 on page 295 in Karatzas&Shreve[KS91]

Theorem 1.1. (Doss(1977)) Suppose that on a certain probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\{\mathcal{F}_t\}$ which satisfies the usual conditions, we have a standard, one-dimensional Brownian motion $(W_t, \mathcal{F}_t, 0 \leq t < \infty)$. Suppose that σ is of class $C^2(\mathbb{R})$, with bounded first and second derivative, and that b is Lipschitz-continuous. Then the one-dimensional stochastic differential equation

$$X_t = X_0 + \int_0^t \{ b(X_s)ds + \frac{1}{2}\sigma(X_s)\sigma'(X_s) \}ds + \int_0^t \sigma(X_s)dW_s \quad (1.1)$$

has a unique, strong solution; this can be written in the form

$$X_t(\omega) = u(W_t(\omega), Y_t(\omega)) \quad (1.2)$$

for a suitable, continuous function $u : \mathbb{R}^2 \rightarrow \mathbb{R}$, and a process Y which solves an ordinary differential equation, for every $\omega \in \Omega$.

Remark 1.1. By integrating Eq 2.38 on page 296 in Karatzas&Shreve[KS91], we see that

$$x = d(y, u) = \int_y^u \frac{d\zeta}{\sigma(\zeta)} \quad (1.3)$$

or $u = u(x, y) = d^{-1}(y, x)$, where $d(., .)$ is the geodesic distance function under the Riemannian metric induced by the inverse of $\sigma(x)$, and d^{-1} is the inverse of d for a fixed value of y . $Y_t(\omega)$ is the solution to

$$\frac{d}{dt} Y_t(\omega) = f(W_t(\omega), Y_t(\omega)) \quad (1.4)$$

i.e. a solution to a *different* ODE for each value of ω .

Theorem 1.2. Assume that $b(x) + \frac{1}{2}\sigma'(x)\sigma(x) = -\frac{1}{2}\sigma^2(x)$, so that

$$dX_t = -\frac{1}{2}\sigma^2(X_t)dt + \sigma(X_t)dW_t \quad (1.5)$$

is the log of a driftless Stock price process. We impose the same conditions as in Theorem 1.1 on b and σ . We further assume that

$$\begin{aligned} C_1 &< \frac{b(x)}{\frac{\partial u(x,y)}{\partial y}} < C_2 \\ 0 < \sigma_1 < \sigma(x) < \sigma_2 < \infty \end{aligned} \quad (1.6)$$

for all x, y , for some real constants C_1, C_2 and σ_1, σ_2 . Then we have the following tail behaviour

$$-\log \mathbb{P}(X_t > x) \sim \frac{d^2(x_0, x)}{2t} \quad (x \nearrow \infty) \quad (1.7)$$

for the distribution function of X_t .

Proof. From the bounds in Eq 1.6, we can strengthen the inequality on $\rho(x, y)$ on page 29 in [KS91] to

$$\frac{1}{C_2} < \rho(x, y) < \frac{1}{C_1} \quad (1.8)$$

Then, using Eq 2.42 on page 297 in [KS91], we see that

$$C_1 t + X_0 < Y_t(\omega) < C_2 t + X_0 \quad (1.9)$$

Eq 1.3 implies that u is monotonically increasing in its second argument, so Eq 1.2 becomes

$$u(W_t(\omega), C_1 t + X_0) \leq X_t(\omega) \leq u(W_t(\omega), C_2 t + X_0) \quad (1.10)$$

The bounds on $\sigma(x)$ in Eq 1.6 ensure that d maps the whole line into itself (see Lamperti [Lam64]). We can re-write this as

$$d^{-1}(C_1 t + X_0, W_t(\omega)) \leq X_t(\omega) \leq d^{-1}(C_2 t + X_0, W_t(\omega)) \quad (1.11)$$

which implies that

$$W_t(\omega) - d(C_1 t + X_0, X_0) \leq d(X_0, X_t(\omega)) \leq W_t(\omega) - d(C_2 t + X_0, X_0)$$

i.e. $d(X_0, X_t)$ is “sandwiched” between two arithmetic Brownian motions. The result is then obtained by applying L'Hôpital's rule to

$$\lim_{x \rightarrow \infty} \frac{\Phi^c(x)}{e^{-(x-\mu)^2/2}} \quad (1.12)$$

where $\Phi^c(x) = 1 - \Phi(x)$, and $\Phi(x)$ is the standard Normal cdf function. From this we see that the drift terms $\mu_{1/2} = d(C_{1/2}t + X_0, X_0)$ are irrelevant as x goes large. \square

1.2 Tail asymptotics for a Dupire local volatility model - the Norris-Stroock Energy functional

Theorem 1.3. (Stroock&Norris, 1991). Let L denote the time-dependent, second order differential operator

$$\begin{aligned} L &= \frac{\partial}{\partial x} (a(t, x) \frac{\partial}{\partial x}) + ab(t, x) \frac{\partial}{\partial x} - \frac{\partial}{\partial x} (a\hat{b}(t, x)) + c(t, x) \\ &= \frac{1}{2}\sigma^2(t, x) \frac{\partial^2}{\partial x^2} - \frac{1}{2}\sigma^2(t, x) \frac{\partial}{\partial x} \end{aligned} \quad (1.13)$$

associated with a **log Stock price process** $dx_t = -\frac{1}{2}\sigma^2(t, x_t)dt + \sigma(t, x_t)dW_t$, W_t a standard Brownian motion, so that

$$\begin{aligned} \frac{1}{2}\sigma^2(t, x) &= a \\ \frac{\partial a}{\partial x} + a(b - \hat{b}) &= -a \\ -\frac{\partial}{\partial x} (a\hat{b}) + c &= 0 \end{aligned} \quad (1.14)$$

where the coefficients a and b are assumed to be differentiable, and a, b and c are measurable functions on $\mathbb{R} \times \mathbb{R}$, and a is uniformly positive and uniformly continuous. We further suppose that there are constants $\lambda \in [1, \infty)$ and $\Lambda \in [0, \infty)$ such that $\lambda^{-1} \leq a(x) \leq \lambda$, $|b|_a^2 + |\hat{b}|_a^2 + |c| \leq \Lambda$, where $|b|_a^2 = \langle b, ab \rangle$. Let $\alpha \in (\frac{1}{2}, 1)$ satisfying $\frac{\alpha^2}{2\alpha-1} > \lambda^2$ be given. Then for all $\lambda \in [1, \infty)$, $\Lambda \in [0, \infty)$ and all $T \in (0, \infty)$, we have the following asymptotic result for the transition density $p(t, x; u; y)$ associated with Eq 1.13

$$\begin{aligned} \lim_{M \nearrow \infty} \inf_{0 < u-t \leq T, x, y \in \mathbb{R}, E(t, x; u, x, y) \geq M} \frac{\log p(t, x; u, y) + E}{E^{\frac{1}{4\alpha-1}}} &= 0 \\ \limsup_{M \nearrow \infty} \sup_{0 < u-t \leq T, x, y \in \mathbb{R}, E(t, x; u, x, y) \geq M} \frac{\log p(t, x; u, y) + E}{\log E} &\leq \frac{N}{4\alpha-2} \end{aligned}$$

where the **Energy functional** is the solution to the variational problem

$$\begin{aligned} E(t, x; u, y) &= \frac{1}{4} \inf_{\gamma \in \Gamma(t, x; u, y)} \int_t^u |\dot{\gamma}(s) - a(b - \hat{b})|_{a^{-1}}^2(s, \gamma_s) ds \\ &= \frac{1}{2} \inf_{\gamma \in \Gamma(t, x; u, y)} \int_t^u \frac{1}{\sigma^2(s, \gamma(s))} |\dot{\gamma}(s) - a(b - \hat{b})|^2 ds \end{aligned} \quad (1.15)$$

where $\Gamma(t, x; u, y) = \{\gamma \in C([t, u], \mathbb{R}) : \gamma_t = x, \gamma_u = y, \text{ and } \int_t^u |\dot{\gamma}_s|^2 ds < \infty\}$.

Remark 1.2. With some pain, Norris&Stroock[NS91] also proved that E is Hölder continuous. They also mention that we have the estimate

$$\frac{\lambda|y-x|^2}{8(u-t)} - \frac{1}{4}\Lambda(u-t) < E(t, x; u, y) < \frac{\lambda|y-x|^2}{2(u-t)} + \frac{1}{2}\Lambda(u-t) \quad (1.16)$$

in terms of the coefficient bounds λ and Λ .

Corollary 1.4. For fixed t, x, u , by Eq 1.16, we see that $E(t, x; u, y)$ can be made arbitrarily large if y is sufficiently large. Thus, Theorem 1.3 implies in particular that for all $\epsilon > 0$, there exists a y such that

$$e^{-\epsilon E^{\frac{1}{4\alpha-1}}} e^{-E} < p(t, x, u, y) < E^{(\frac{N}{4\alpha-2}+\epsilon)} e^{-E} \quad (1.17)$$

which implies the following tail behaviour on logarithmic scale

$$-\log p(t, x; u, y) \sim E(t, x; u, y) \quad (|y| \nearrow \infty) \quad (1.18)$$

Note the similarity of the Energy functional to the *geodesic distance* discussed in Appendix A, which we can write as

$$\frac{1}{2}d^2(x; y) = \frac{1}{4} \inf_{\gamma \in \Gamma(0, x; 1, y)} \int_0^t |\dot{\gamma}(s)|_{a^{-1}}^2(s, \gamma_s) ds \quad (1.19)$$

The factor of $\frac{1}{4}$ appears instead of the usual $\frac{1}{2}$ because, in Norris and Stroock's definition, $a(x) = \frac{1}{2}\sigma^2(x)$ as opposed to $\sigma^2(x)$. $E(t, x; u, y)$ and $d^2(x, y)/2(u - t)$ only coincide in the time-independent case when $b = \hat{b}$.

Under sufficient smoothness assumptions, computing E amounts to a calculus of variations problem, where we have to minimize the functional

$$\int_0^1 L(x, \dot{x}, t) dt \quad (1.20)$$

subject to $x(0) = a, x(1) = b$. The optimal solution is obtained by solving the *Euler-Lagrange equation*

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \quad (1.21)$$

In the time-independent case, we can write Eq 1.20 as

$$L(x, y, t) = m(x)(y - \mu(x))^2 \quad (1.22)$$

where $m = 1/2\sigma^2(x)$, and $\mu(x) = a(x)(b(x) - \hat{b})$, so Eq 1.21 becomes

$$\frac{d}{dt} \left[2m(x) \left(\frac{dx}{dt} - \mu(x) \right) \right] = m'(x) \left(\frac{dx}{dt} - \mu(x) \right)^2 - 2m(x) \left(\frac{dx}{dt} - \mu(x) \right) \mu'(x) \quad (1.23)$$

which is a non-linear second order differential equation for $x(t)$ (see numerics overleaf).

1.3 Large-strike behaviour of the Implied volatility smile

Theorem 1.3 is similar in flavour to Theorem 2 in Busca et al.[BBF02], which shows, under certain technical conditions, that the implied volatility of European put/call options have the large-strike behaviour

$$\lim_{x \rightarrow \pm\infty} \hat{\sigma}(x, T) = \left(\frac{1}{T-t} \int_t^T \hat{\sigma}_\pm^2(u) du \right)^{1/2} \quad (1.24)$$

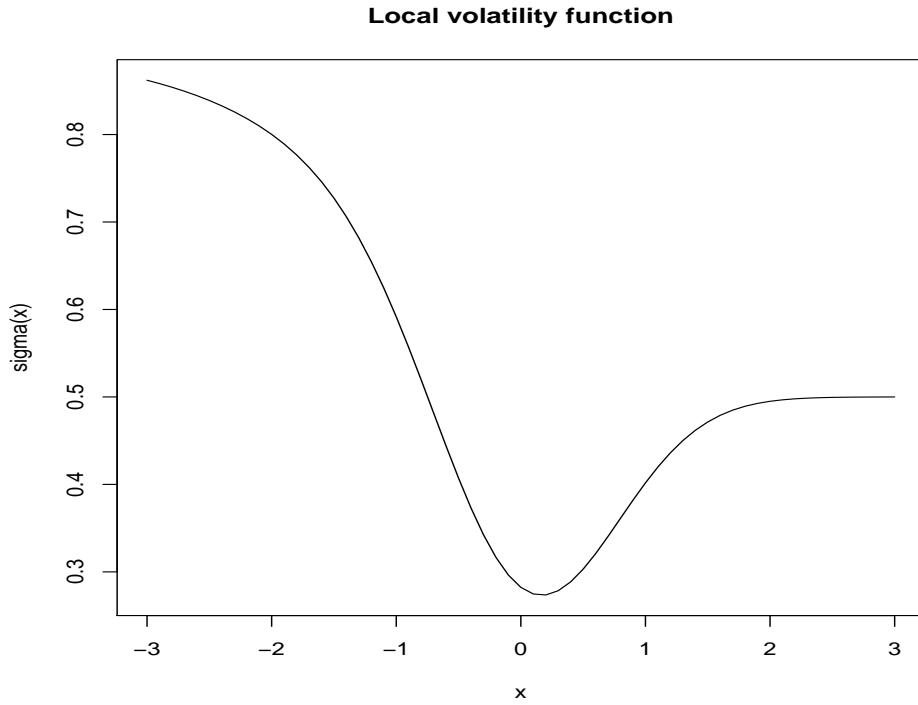


Figure 1: The local volatility function $\sigma(x) = \frac{1}{2} - 0.1e^{1-x^2} + 0.4e^{-2e^x}$

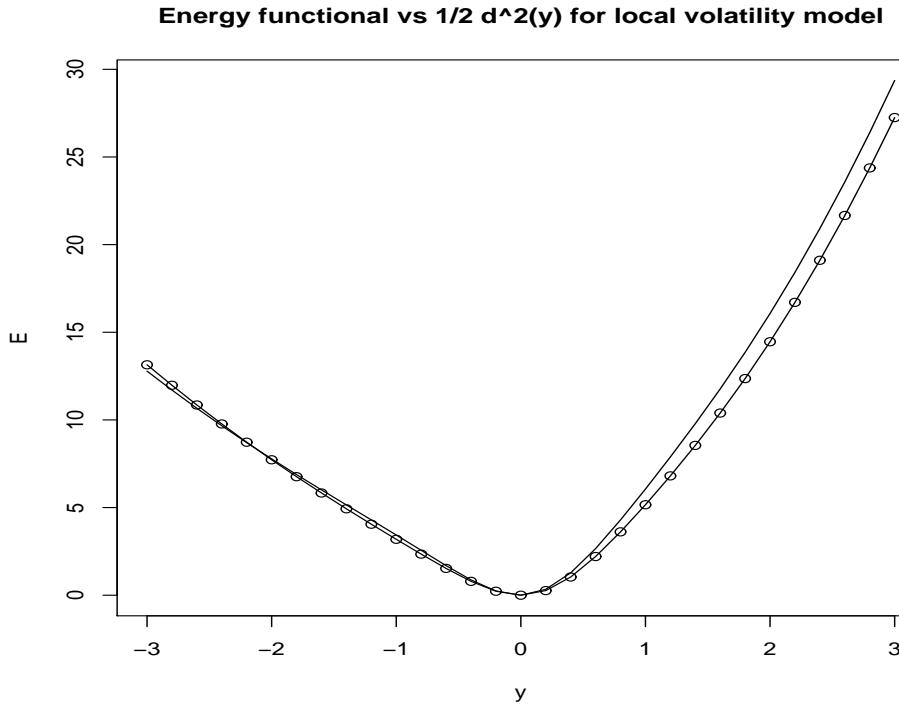


Figure 2: Here we have plotted the Energy functional vs $\frac{1}{2}d^2(x, y)$ for the diffusion process above with $x = 0, T = 1$, where $d(y) = \int_0^y \frac{d\zeta}{\sigma(\zeta)}$. The curve with points is $\frac{1}{2}d^2(y)$. Remember that the higher the value, the lower the weight in the tail. This was computed in Maple by numerically solving the Euler-Lagrange equation.

where

$$\sigma_{\pm}(t) = \lim_{x \rightarrow \pm\infty} \sigma(t, x) \quad (1.25)$$

We now sharpen this result by taking into account how $\sigma(x)$ behaves at *all* values of x , via the Energy functional.

Theorem 1.5. *For the log Stock price process associated with the operator L in , we have the following asymptotic behaviour for the implied volatility $I(k)$ of European options of log-moneyness k , expiring at u*

$$\frac{I^2(k)}{k} \sim \frac{k}{2E(t, x_0; u, x_0 + k)} \quad (1.26)$$

as $k \rightarrow \infty$. $\psi : [0, \infty] \rightarrow [0, 2]$ is given by $\psi(x) = 2 - 4[\sqrt{x^2 + x} - x]$. This means that $I^2(k)$ behaves **sub-linearly** for large k , because the bounds in 1.16 imply that the right hand side tends to zero as $k \rightarrow \infty$.

Proof. This follows from Eq (v) in the right-tail-wing formula in Benaim&Friz[BF06 I]. \square

1.4 Tail asymptotics for a stopped CEV process

In this section, we consider the *constant elasticity of variance*(CEV) process of Cox[Cox75]:

$$dX_t = \delta X_t^{1+\beta} dW_t \quad (1.27)$$

with $\beta < 0$, where W_t is a standard Brownian motion. The CEV process can be obtained as a simple transformation of a squared Bessel process (see Revuz&Yor[RY91] and Linetsky[Lin04]), which is why it gives rise to analytical formulae for many derivatives of interest . Zero is an absorbing barrier for $\beta \in (-\frac{1}{2}, 0)$, and a reflecting barrier for $\beta < -\frac{1}{2}$, which is why we **stop** the process at the first passage time to zero. Under this specification, bankruptcy is a consequence of the Stock price falling ever closer to zero, in contrast to a Poisson default process. A closed-form expression for the risk-neutral probability of absorption (i.e. bankruptcy) is given in e.g. Davydov&Linetsky[DavLin01]. We refer the reader to Atlan&Leblanc[AL05] for further details and references.

Theorem 1.6. *If X_t follows the CEV process above, so that the origin specified as a **killing** boundary if $\beta < -\frac{1}{2}$, then the right tail of the transition density $p(T; x_0, x)$ for X_T has the following asymptotic behaviour*

$$-\log p(T; x_0, x) \sim \frac{x^{2|\beta|}}{2\delta^2\beta^2 T}$$

Proof. The transition density for the stopped CEV process is given in e.g. Davydov&Linetsky[DavLin01] as

$$p(T; x_0, x) = \frac{x^{-2\beta-3/2} x_0^{1/2}}{\delta^2 |\beta| T} \exp\left(-\frac{x_0^{-2\beta} + x^{-2\beta}}{2\delta^2 \beta^2 T}\right) I_{\nu}\left(\frac{x_0^{-\beta} x^{-\beta}}{\delta^2 \beta^2 T}\right) \quad (1.28)$$

where $\nu = \frac{1}{2|\beta|}$, and $I_\nu(\cdot)$ denotes the modified Bessel function of order ν . Note that this density will integrate to less than one, because of the non-zero probability of absorption. Then we just use the asymptotic relation

$$I_\nu(z) \sim \frac{1}{\sqrt{2\pi z}} e^z \quad (1.29)$$

for $z \gg |\nu^2 - \frac{1}{4}|$, then take logs. \square

Theorem 1.7. *If X_t follows the constant elasticity of variance(CEV) process above, then we have the following asymptotic behaviour for the implied volatility $I(k)$ of standard European put/call options of log-moneyness k*

$$\frac{I^2(k)}{k} \sim k (x_0 e^k)^{-2|\beta|} \cdot \delta^2 \beta^2 \quad (1.30)$$

as $k \rightarrow \infty$. $\psi : [0, \infty] \rightarrow [0, 2]$ is given by $\psi(x) = 2 - 4[\sqrt{x^2 + x} - x]$. This means that $I^2(k)$ behaves **sub-linearly**, because the right hand side tends to zero as $k \rightarrow \infty$. It also means that there is no term-structure of implied volatility in the right wing, because the right-hand side is independent of T .

Proof. Using Eq 1.28, the density of the risk-neutral return $\log \frac{X_T}{X_0}$ is given by

$$f(k) = x_0 e^k \frac{(x_0 e^k)^{-2\beta-3/2} x_0^{1/2}}{\delta^2 |\beta| T} \exp\left(-\frac{x_0^{-2\beta} + (x_0 e^k)^{-2\beta}}{2\delta^2 \beta^2 T}\right) I_\nu\left(\frac{x_0^{-\beta} (x_0 e^k)^{-\beta}}{\delta^2 \beta^2 T}\right) \quad (1.31)$$

Then

$$\log f(k) \sim \frac{(x_0 e^k)^{-2\beta}}{2\delta^2 \beta^2 T} \quad (1.32)$$

The result then follows from Eq (v) in the right-tail-wing formula in Benaim&Friz[BF06 I]. \square

2 Tail asymptotics for a diffusion process evaluated at an independent stochastic clock

2.1 The general time-changed diffusion

We now assume that the reference entity's Stock price process S_t is a one-dimensional diffusion process $(X_t)_{t \geq 0}$ evaluated at a random time given by a increasing, right continuous process $\tau(\omega, t)$, which is *independent* of X_t ; thus

$$\begin{aligned} S_t &= X_{\tau(t)} \\ dX_t &= \sigma(X_t) dW_t \end{aligned} \quad (2.1)$$

with $\sigma(x)$ bounded between two positive constants, and uniformly Hölder continuous on \mathbb{R}^+ . $\tau(T)$ may be continuous or purely discontinuous (e.g. an increasing Lévy process); see Geman,Madan&Yor[GMY02] and Cont&Tankov[CT04] for further discussion on this point.

One way to represent the distribution of the stopping time $\tau_y = \inf\{t \geq 0 : X_t = y \in (0, \infty)\}$ is to express its Laplace transform $\mathbb{E}^x(e^{-\lambda\tau_y} 1_{\{\tau_y < \infty\}})$ in terms of the increasing and decreasing solutions of the *Sturm-Liouville* equation

$$(\mathcal{A}f)(x) - \lambda f(x) = 0 \quad (2.2)$$

for $\lambda > 0$, (see Davydov&Linetsky[DavLin01], Borodin&Salminen[BS96] and Karlin&Taylor[KT81]). Specifically, we have

$$\begin{aligned} \mathbb{E}_x(e^{-\lambda\tau_L} 1_{\{\tau_L < \infty\}}) &= \frac{\phi_\lambda(x)}{\phi_\lambda(L)}, \quad x \geq L \\ \mathbb{E}_x(e^{-\lambda\tau_U} 1_{\{\tau_U < \infty\}}) &= \frac{\psi_\lambda(x)}{\psi_\lambda(U)}, \quad x \leq U \end{aligned} \quad (2.3)$$

Under the conditions have we placed on $\sigma(x)$, 0 and ∞ are *natural boundaries*, and $\psi_\lambda(0_+) = 0$, $\phi_\lambda(0_+) = \infty$, $\lim_{x \rightarrow \infty} \psi_\lambda(x) = \infty$, $\lim_{x \rightarrow \infty} \phi_\lambda(x) = 0$.

Remark 2.1. $\phi_\lambda(x)$ and $\psi_\lambda(x)$ are called **fundamental solutions** of the ODE in Eq 2.2. They are linearly independent, and all solutions to Eq 2.2 can be expressed as their linear combination.

By the Feynman-Kac formula, the solution to the Cauchy problem

$$\begin{aligned} -\frac{\partial f}{\partial t} &= \mathcal{A}f \\ f(T, x) &= \psi_\lambda(x) \end{aligned} \quad (2.4)$$

has the stochastic representation

$$f(t, x) = \mathbb{E}^{t,x}(\psi_\lambda(X_T)) = \psi_\lambda(x) e^{\lambda(T-t)} \quad (2.5)$$

and similarly for ϕ_λ , because zero is a natural boundary. For the subordinated S_t process, by conditioning on the independent $\tau(T)$ at time zero, we observe that

$$\mathbb{E}(e^{\lambda\tau(T)}) = \mathbb{E}^{0,S_0}\left(\frac{\psi_\lambda(S_T)}{\psi_\lambda(S_0)}\right) = \mathbb{E}^{0,S_0}\left(\frac{\phi_\lambda(S_T)}{\phi_\lambda(S_0)}\right) \quad (2.6)$$

Remark 2.2. If X_t is a standard Brownian motion, then the eigenfunctions are given by $e^{\pm\sqrt{2\lambda}x}$.

2.2 Tail asymptotics for a time-changed CEV process

We now consider the right tail asymptotics for the transition densities of stopped CEV diffusion, but evaluated at an independent stochastic clock. Adding this stochastic volatility component means that the smile effect is less correlated to the probability of default, and it means that the right tail decays significantly slower (this is made precise below). We do not correlate the time-change with the CEV process, because the leverage effect is more than adequately explained by the power law in the CEV diffusion coefficient.

Theorem 2.1. *For a CEV process evaluated at an independent random time $\tau(t)$, we have the following one-to-one correspondence between exponential moment explosions for $\tau(\omega, T)$ and $S_T = X_{\tau(\omega, t)}(\omega)$*

$$\begin{aligned}\lambda^*(T) &= \sup\{\lambda \in \mathbb{R}^+ : \mathbb{E}e^{\lambda\tau(T)} < \infty\} \\ &= \sup\{\lambda \in \mathbb{R}^+ : \mathbb{E}(\exp(\frac{\sqrt{2\lambda}}{\delta|\beta|} S_T^{-\beta})) < \infty\}\end{aligned}\quad (2.7)$$

if $\lambda^*(T) \in (0, \infty)$, which ensures that the tails of $S_T = X_{\tau(T)}$ are realistically fat.

Corollary 2.2. *For any random variable X , a simple Chebyshev argument gives rise to the following result (see Benaim&Friz[BF06 II])*

$$-\limsup_{x \rightarrow \infty} \frac{\log \mathbb{P}(X > x)}{x} = q^* = \sup\{q : \mathbb{E}e^{qX} < \infty\}\quad (2.8)$$

so we have the following estimate for the tail behaviour of the distribution function of S_T

$$-\limsup_{x \rightarrow \infty} \frac{\log \mathbb{P}(\frac{1}{\delta|\beta|} S_T^{-\beta} > x)}{x} = \sqrt{2\lambda^*(T)}\quad (2.9)$$

Remark 2.3. Corollary 2.2 could be applied to extrapolate the prices of digital call options to large strikes, where PDE and Monte Carlo methods break down. It may be possible to combine this with the tail asymptotics for the integrated variance under the Heston model reported in section 6.3 of Benaim&Friz[BF06 II], to prove that the \limsup here is indeed a limit. We suspect that this is the case, and if so, one could then invoke the right-wing-tail formula in Benaim&Friz[BF06 I]. However, this appears to be difficult because we need to ignore the $\frac{1}{\sqrt{2\pi z}}$ term in Eq 1.29 in order to prove Theorem 2.1.

Proof. (of Theorem 2.1) The CEV process violates the condition on the boundedness of the diffusion coefficient in section 1.1, but the first equality in Eq 2.6 still holds. In this case, $\psi_\lambda(S)$ is given by

$$\psi_\lambda(S) = S^{\frac{1}{2}} I_v(\sqrt{2\lambda}) z\quad (2.10)$$

where

$$z = z(S) = \frac{1}{\delta|\beta|} S^{-\beta}\quad (2.11)$$

and $\nu = \frac{1}{2|\beta|}$. $I_\nu(\cdot)$ denotes the modified Bessel function (see Davydov&Linetsky[DavLin01]). Now, using the asymptotic relation in Eq 1.29, we see that for all $\epsilon > 0$, there exists a $K_1(\epsilon)$ such that

$$\int_{K_1}^\infty \psi_\lambda(S) p(S) dS < (1 + \epsilon) \int_{K_1}^\infty \frac{S^{\frac{1}{2}}}{\sqrt{2\pi \sqrt{2\lambda} z(S)}} e^{\sqrt{2\lambda} z(S)} p(S) dS\quad (2.12)$$

where $p(S)$ is the transition density for S_T , and there exists a $K_2(\epsilon)$ such that $(1 + \epsilon) \frac{S^{\frac{1}{2}}}{\sqrt{2\pi \sqrt{2\lambda} z(S)}} e^{\sqrt{2\lambda} z(S)} < e^{\sqrt{2\lambda} z(S)(1+\epsilon)}$ for all $S > K_2(\epsilon)$. Taking $K(\epsilon) =$

$\max(K_1(\epsilon), K_2(\epsilon))$ and remembering that $\psi_\lambda(S)$ is monotonically increasing in S , we see that

$$\mathbb{E}(\psi_\lambda(S_T)) = \infty \Rightarrow \mathbb{E}\left[\exp(\sqrt{2\lambda}z(S_T)(1+\epsilon))\right] = \mathbb{E}\exp\left(\frac{\sqrt{2\lambda}}{\delta|\beta|}S_T^{-\beta}(1+\epsilon)\right) = \infty$$

We proceed similarly for the lower bound. \square

Example 2.1. If $\tau(t)$ is obtained as the integral of the Cox-Ingersoll-Ross(CIR) square root diffusion process $dv_t = \kappa(\theta - v_t)dt + \sigma\sqrt{v_t}dW_t^2$, $\sigma, v_0 > 0$, then the critical value $\lambda^*(T)$ can be computed numerically, see e.g. Proposition 3.1 in Andersen&Piterbarg[AP05] or section 6.3 of Benaim&Friz[BF06 II].

Remark 2.4. For an uncorrelated SABR model where the time-change is given by the exponential functional of Brownian motion $A_t^{(\mu)} = \int_0^t e^{2(B_s+\mu s)}ds$ with $\mu = -\frac{1}{2}$, $\mathbb{E}e^{\lambda\tau(T)} = \infty$ for all $\lambda > 0, T > 0$ (see Jourdain[Jour04]), so 2.1 only tells us that

$$\mathbb{E}\left(\exp\left(\frac{\sqrt{2\lambda}}{\delta|\beta|}S_T^{-\beta}\right)\right) = \infty \quad (2.13)$$

for all λ . The difficulty with the SABR model lies in characterizing the tail behaviour of $\tau(t)$. At the present time, we have only been able to establish the Gaussian bounds

$$\exp\left[-\frac{x^2(1+\epsilon)}{\frac{8}{3}t}\right] \leq \mathbb{P}\left(\frac{\tau(T)}{T} > x\right) \leq \exp\left[-\frac{x^2(1+\epsilon)}{8t}\right] \quad (2.14)$$

for all $\epsilon > 0$, if x is sufficiently large, using Jensen's inequality for the lower bound which is slightly less trivial. There is a closed-form expression for the joint density of $(\log S_t, v_t)$ under the SABR model with $\beta = 1, \rho = 0$, originally obtained by McKean (see Hagan et al.[HLW04], and Matsumoto&Yor[MatsYor05 I], [MatsYor05 II]).

Remark 2.5. In the case when $\mathbb{E}e^{\lambda\tau(T)} < \infty$ for all $\lambda > 0$, it may should be possible to derive a sharper result than 2.1, using e.g. **Kasahara's Tauberian theorem** (see Bingham et al.[BGT87]) to relate the tail behaviour of $\tau(T)$ and the tail behaviour of $\frac{S_T^{-\beta}}{\delta|\beta|}$. However, this case is of little practical use in Mathematical finance, because we would typically add a stochastic volatility/time-change component to fatten the tails of the risk-neutral densities, so that some of the moments of the terminal Stock price explode.

Remark 2.6. Conversely, it is also possible to derive bounds on the moment generating function, and hence the tails, for a general time-changed diffusion when $\lambda^*(T) \in (0, \infty)$, using Doss's theorem on pg 295 in Karatzas&Shreve[KS91]. However, these are somewhat cumbersome because they involve bounding constants on the drift and diffusion coefficients, so we omit the details

3 Calibrating a time-changed diffusion model: the Carr-Lee inverse problem of extracting the law of the time-change from a single-maturity smile

We now consider the inverse calibration problem of extracting the law of the time-change $\tau(T)$ in Eq 2.1, given the law of $S_T = X_{\tau(T)}$, when $\sigma(x)$ is known. We would typically obtain the law of S_T from the *observed* implied volatility smile at maturity T . This problem was considered at length for the case when the X_t process is geometric Brownian motion (i.e. $\sigma(x) = 1$) by Carr&Lee[CL04], and later by Friz&Gatheral[FrizGath05].

In principle, we can extend the Carr-Lee approach to arbitrary $\sigma(x)$, using Eq 2.6 to reverse engineer the characteristic function of $\tau(T)$ from the prices of so-called *eigenfunction contracts* as $\mathbb{E}(e^{i\theta\tau(T)}) = \mathbb{E}(\frac{\psi_{i\theta}(S_T)}{\psi_{i\theta}(S_0)})$ (for $\theta \in \mathbb{R}$), if the Expectation on the right-hand side exists. We could span the complex-valued eigenfunction contract paying $\frac{\psi_{i\theta}(S_T)}{\psi_{i\theta}(S_0)}$ with a portfolio of standard call and put options, using the well known decomposition of a twice differentiable payoff given in e.g. the Appendix of Carr&Madan[CM98]. We could then compute the distribution function of S_T , using Lévy's inversion theorem. For $\tau(t)$ continuous, the most important choice for $\sigma(x)$ is the Carr-Lee case with $\sigma(x) = 1$, because then the time-change agrees with the quadratic variation of the log-returns process (by the Dambis-Dubins Schwarz result), which is the underlying for liquidly-traded volatility derivatives. Nevertheless, calibrating the law of $\tau(T)$ in the general case is important if we wish to build a model which is consistent not only with the observed prices of European options, but also with barrier and forward starting options. The Dupire local volatility model discussed in section 1.4 typically falls short in this respect, because it internalizes the wrong dependence structure.

When the underlying diffusion process in Eq 2.1 is a CEV process with $\sigma(S) = S^{1+\beta}$, $\beta \in (0, 1)$, then setting $\lambda = i\theta$ for $\theta \in \mathbb{R}$, Eq 2.6 becomes

$$\mathbb{E}\left[\frac{S^{\frac{1}{2}} I_v(\sqrt{(2i\theta)} z(S_T))}{S_0^{\frac{1}{2}} I_v(\sqrt{(2i\theta)} z(S_0))}\right] = \mathbb{E}(e^{i\theta\tau(T)}) \quad (3.1)$$

By a very similar argument to Theorem 2.1, we see that

$$\begin{aligned} & \mathbb{E}(e^{i\theta\tau(T)}) \\ = & \mathbb{E}\left[\frac{S_T^{\frac{1}{2}} I_v(\sqrt{(2i\theta)} z(S_T))}{S_0^{\frac{1}{2}} I_v(\sqrt{(2i\theta)} z(S_0))}\right] < \infty \\ \iff & \mathbb{E}\left[\exp\left(\frac{\sqrt{2i\theta}}{|\beta|}(S_T^{-\beta} - S_0^{-\beta})\right)\right] \\ = & \mathbb{E}\left[\exp\left(\frac{(1+i)\sqrt{\theta}}{|\beta|}(S_T^{-\beta} - S_0^{-\beta})\right)\right] \\ = & \int_0^\infty \exp\left(\frac{(1+i)\sqrt{\theta}}{|\beta|}(S^{-\beta} - S_0^{-\beta})\right) p(T; S_0, S) dS < \infty \quad (3.2) \end{aligned}$$

However, the Lebesgue integral in the last line will not exist if the integrand is not *absolutely* integrable, so the restriction that

$$\int_0^\infty \exp\left(\frac{\sqrt{\theta}}{|\beta|}(S^{-\beta} - S_0^{-\beta})\right) p(T; S_0, S) dS = \mathbb{E}\left[\exp\left(\frac{\sqrt{\theta}}{|\beta|}(S_T^{-\beta} - S_0^{-\beta})\right)\right] < \infty \quad (3.3)$$

provides a necessary condition on the tail behaviour of S_T , so we can compute the value of $\mathbb{E}(e^{i\theta\tau(T)})$ for all $\theta \in \mathbb{R}$. This means that

$$-\limsup_{x \rightarrow \infty} \frac{\log \mathbb{P}(\tau > x)}{x} = \infty \quad (3.4)$$

so the tails of the distribution function of $\tau(T)$ have to decay faster than any exponential. This condition is violated by the Heston and SABR models, and may well explain why the Friz-Gatheral Moore-Penrose regularization scheme in [FrizGath05] performs so badly for high volatility of the instantaneous variance under the Heston model. Our extended Carr-Lee methodology fails in these cases because, for certain values of θ in Eq 3.2, we go outside the strip of regularity of the characteristic function of $(S_T^{-\beta} - S_0^{-\beta})/|\beta|$, where it cannot be represented as a Fourier integral. It is easily shown that same restriction applies for the original Carr-Lee framework with $\sigma(x) = 1$. The only other way to circumvent this problem would be to compute $\Psi(\theta) = \mathbb{E}(e^{i\theta\tau(T)})$ on a domain $G \subset \mathbb{C}$, and then use *analytic continuation*, by computing a truncated approximation to the power series representation for Ψ .

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Appendix: The Varadhan density estimate, and the geodesic distance

For a diffusion process with generator $\frac{1}{2} \sum a_{ij}(x) \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j}$, where a_{ij} satisfies uniform Hölder and ellipticity conditions, the Varadhan[Var67PDE] density estimate states that

$$\lim_{t \rightarrow 0} -2t \log p(t, x, y) = d^2(x, y) \quad (\text{B. 1})$$

uniformly over x and y such that $d(x, y)$ is bounded, where

$$d(x, y) = \inf_{f: f(0)=x, f(1)=y} \int_0^1 \sqrt{\left(\sum_{i,j=1}^d g_{ij} \frac{df^i}{dt} \frac{df^j}{dt} \right)} dt$$

(for \mathbb{R}^d -valued $f \in C[0, 1]$) is the shortest (or *geodesic*) distance from x to y on a Riemannian manifold with metric $g_{ij} = (a^{ij})^{-1}$, which is the inverse of

$a = \sigma\sigma^T$. Under sufficient regularity conditions, we can use the Euler-Lagrange equation from calculus of variations theory to show that the optimal path satisfies the geodesic equation

$$\frac{d^2 f^\mu}{d\tau^2} + \Gamma_{\mu\beta}^\alpha \frac{df^\mu}{dt} \frac{df^\beta}{dt} = 0 \quad (\text{B. 2})$$

where the Christoffel symbol $\Gamma_{\mu\beta}^\alpha = \frac{1}{2}g^{\alpha\nu}(g_{\nu\mu,\beta} + g_{\nu\beta,\mu} - g_{\mu\beta,\nu}) = \frac{1}{2}a^{\alpha\nu}(g_{\nu\mu,\beta} + g_{\nu\beta,\mu} - g_{\mu\beta,\nu})$ is the α -th component of $\frac{\partial \vec{e}_\alpha}{\partial x^\beta}$. Geodesics **parallel transport** their own tangent vector $\vec{U} = \frac{df}{d\tau}$ i.e. the covariant derivative $U^\beta U_{;\beta}^\alpha = 0$). This implies that the scalar product

$$g(\vec{V}, \vec{V}) = \sum_{i,j=1}^d g_{ij} \frac{df^i}{dt} \frac{df^j}{dt} \quad (\text{B. 3})$$

is constant along geodesic paths. The geodesic length $d(x, y)$ is *independent* of the parametrization of the curve. However, the following functional

$$I(x, y) = \frac{1}{2} \inf_{f: f(0)=x, f(1)=y} \int_0^t \sum_{i,j=1}^d g_{ij} \frac{df^i}{dt} \frac{df^j}{dt} dt$$

which crops up in the Wentzell-Freidlin theory of large deviations for SDEs(see Varadhan[Var84]), *does* depend on the parametrization. However, for a class of parametrizations all representing the same curve in \mathbb{R}^d , $I(x, y)$ is minimized when the curve is parametrized so as to cover equal lengths in equal times i.e. $I(x, y) = \frac{1}{2} \int_0^t c dt = \frac{1}{2}ct$, $d(x, y) = \int_0^t \sqrt{c} dt = \sqrt{c}t$ so that $I(x, y) = \frac{1}{2}d^2(x, y)/t$ (see Varadhan[Var67Prob]).